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ABSTRACT

From a decision theoretic viewpoint, a general coefficient (delta) for tests is derived. The coefficient is applied to three kinds of decision situations. The first situation involves a true score estimated by a function of the observed score of a subject on a test (point estimation). Using the squared error loss function and Kelley's formula for estimating the true score, it is shown that delta equals the reliability coefficient from classical test theory. The second situation involves the observed scores split into more than two categories after which different decisions are made for the categories (multiple decision). The general form of the coefficient is derived, and two loss functions suited to multiple decision situations are described. It is shown that, for the loss function specifying constant losses for the various combinations of categories on the true and observed scores, the coefficient can be computed under the assumptions of the beta-binomial model. The third situation involves splitting of the observed scores into only two categories and making different decisions for the categories (dichotomous decisions). Finally, it is shown that for a linear loss function and Kelley's formula for the regression of the true score on the observed score, the coefficient equals the reliability coefficient of classical test theory. (TJH)

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Coefficients for Tests from a Decision Theoretic Point of View

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Abstract

From a decision theoretic point of view a general coefficient δ for tests is derived. The coefficient is applied to three kinds of decision situations. First, the situation is considered, in which a true score is estimated by a function of the observed score of a subject on a test (point estimation). Using the squared error loss function and Kelley's formula for estimating the true score, it is shown that δ equals the reliability coefficient from classical test theory. Second, the situation is considered in which the observed scores are split into more than two categories and different decisions are made for the categories (multiple decision). The general form of the coefficient is derived, and two loss functions suited to multiple decision situations are described. It is shown that for the loss function specifying constant losses for the various combinations of categories on the true and on the observed scores the coefficient can be computed under the assumptions of the beta-binomial model. Third, the situation is considered in which the observed scores are split into only two categories and different decisions made for the two categories (dichotomous decisions). Using a loss function that specifies constant losses for the combinations of the categories on the true and observed score, and the assumption of an increasing regression function of t on x the coefficient δ equals Loevinger's coefficient H between true and observed scores. The coeficient can be computed under the assumption of the beta-binomial model. Finally, it is shown that for a linear loss function and Kelley's formula for the regression of the true score on the observed score the coefficient equals the reliability coefficient of classical test theory.

Decision Situations in Testing

Psychological and educational tests can be considered as instruments for making decisions. In research the decision problem is mostly the estimation of a true score t, using a function of the observed score x of a subject. The problem can be described as point estimation of the parameter t of this subject; the possible actions of the decision maker are all possible estimated values for the true score (Ferguson, 1967, p.11). The decision rule is in this case the procedure used for estimating the true score. For example, in classical test theory the following decision rules are used: estimate the true score by the value of the observed score or use Kelley's formula for estimating the true score (Lord & Novick, 1968, p. 63).

In applied fields the decision problem is to classify subjects into different categories. The possible actions of the decision maker are the different categories into which subjects are classified. With more than two categories the problem is called a multiple decision problem (Ferguson, 1967, p.10). A very common decision problem is to classify subjects into only two categories: accepted or rejected, for instance, pass-fail decisions in education, or acceptance-rejection decisions for applicants for jobs or for special treatments such as psychotherapy or remedial teaching. These problems will be called dichotomous decision problems (Mellenbergh, Koppelaar, & Van der Linden, 1978).

For solving decision problems a loss function L(t,x) is needed that depends on the true score and the observed score; this function specifies

the loss of the decision maker using a certain decision rule. For instance, in dichotomous decision problems the loss function specifies the loss of the decision maker for "accepted, suitable," "accepted, not suitable," "rejected, suitable," and "rejected, not suitable" subjects. The risk of a decision rule is, for given t, the expected value of the loss with respect to the distribution of x (Ferguson, 1967, p.7). The Bayes risk is the expected value of the risk with respect to the distribution of t (Ferguson, 1967, p.31). The result is that the Bayes risk can be considered as the expected value of the loss with respect to the joint distribution of the random variables t and x in a given population of subjects: $R = E\ L(t,x)$. Shortly, the Bayes risk will be called the risk.

Attention has recently been paid in psychometrics to dichotomous decision situations, especially for mastery decisions in criterion-referenced measurement (Hambleton & Novick, 1973; Huyhn, 1976b; Meskauskas 1976). Meskauskas (1976) distinguished between State and Continuum models for mastery decisions. In State models the true score is considered an all or none variable, which represents mastery or non-mastery of the subject matter. In Continuum models the true score is considered a continuous variable which represents the degree of mastery of the subject matter. Macready and Dayton (1977) have developed State models from a decision theoretic point of view; Huyhn (1976b) has considered mastery decisions with a continuous true score. Huyhn (1976a, 1976c) has also described coefficients for tests in dichotomous decision situations. In this article a general strategy for constructing coefficients for tests from a decision theoretic point of view is

described; the true score is considered a continuous variable.

Coefficients for Tests

In decision theory the risk can be considered as an index of the quality of the decision: the smaller the risk the better the decision rule.

Therefore, the risk is a proper starting-point for deriving a decision-oriented coefficient for tests. Two disadvantages have to be removed. First, in test theory it is more conventional to define indices of which the scale has a direction opposite to that in which the risk is represented. Second, the range of possible values for the risk can be different from the standard interval [0,1], whereas in test theory indices are nearly always defined on this interval. Both disadvantages can be removed by defining two reference points, say R_c and R_n, that have a decision theoretic interpretation: R_c and R_n are the risks in the situation in which the test contains complete, respectively no information about the true score. Using these two reference points an interpretable index for decision situations is:

$$\delta = 1 - (R - R_c)/(R_n - R_c) = (R_n - R)/(R_n - R_c). \qquad [1]$$

Owing to this linear transformation the scale of δ has a direction opposite to that of the scale in which R is represented. Furthermore, δ will be in the conventional interval from 0 to 1 whenever R is in the interval from R_c to R_n. The true score t is defined on the closed interval from 0 to 1; the distribution of t is g(t). It is assumed that the test is composed of n items that are scored 0 or 1; the observed score x is the sum of the item scores. In this case x is defined on the set of integers: 0, 1, 2, ... n; the distribution of x is h(x). If k(t,x) is the bivariate distribution of x and t the risk is:

$$R = E\, L(t,x) = \sum_{x=0}^{n} \int_0^1 L(t,x)\, k(t,x)\, dt. \qquad [2]$$

R_n is defined as the risk in the situation in which, for fixed distributions h(x) and g(t), the observed score contains no information about the true score. This notion is formalized by the statement that x and t are distributed independently:

$$k(t,x) = g(t)\ h(x). \qquad [3]$$

R_c is defined as the risk in the situation in which, for a fixed distribution h(x), the observed score contains complete information about the true score. This notion is formalized by the statement that t is an increasing function of x which maps the set of integers 0, 1, 2, ...n into the closed interval from 0 to 1.

In general R is not necessarily in the interval from R_c to R_n, and consequently δ not in the interval from 0 to 1. The reason is that R, R_c, and R_n depend on the loss function and on the bivariate distribution of t and x. R_c and R_n, however, can be considered as rather interpretable points for the risk. R_c is the risk when the test contains complete information about the true score; R_n the risk when the test contains no information about the true score. In the remainder of this article δ is derived for some cases. In some very important cases the risk is in the interval from R_c to R_n; therefore, δ has a clear interpretation: a value of 0 means that the test is worthless and a value of 1 that the test is perfect for the decision situation.

Point Estimation

When the decision problem is estimating the parameter t, one must seek a function p(x) that is a suitable point estimator for t. The function p(x) is the decision rule: it gives for each possible value of x an estimated value p(x) for t. Here, interest is not in finding point estimators with recognized statistical properties, but in the assessment of the decision theoretic qualities of given point estimators.

For evaluating a point estimator, it is necessary to specify a loss function that provides a weighing of possible errors of estimation. Defining an error of estimation as the difference between the parameter t and the function p(x) used for estimating t, the loss is often a function of p(x) and t:

$$L(t,x) = L(p(x) - t). \qquad [4]$$

Substitution of this loss function into Equation 2 yields a general form for the risk of a point estimator:

$$R = \sum_{x=0}^{n} \int_{0}^{1} L(p(x) - t)\, k(t,x)\, dt. \qquad [5]$$

This risk will be the starting-point for the evaluation of a given point estimator with the decision-oriented coefficient δ given in

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Equation 1. First, the procedure is described for the regression function of t on x; this regression function $E(t|x)$ can be used as a point estimator for the parameter t. Second, the result of the procedure is shown for the linear regression function of t on x, better known as Kelley's formula, for estimating the true score. In both cases, the loss function is the squared error loss function

$$L(t,x) = (p(x) - t)^2 . \qquad [6]$$

which is well-known from its use in statistics (Novick & Jackson, 1974, Chap. 1).

For the regression function $E(t|x)$ and loss function Equation 6 the risk is:

$$R = \sum_{x=0}^{n} \int_0^1 (E(t|x) - t)^2 \, k(t,x) \, dt. \qquad [7]$$

In the situation in which x and t are stochastically independent Equation 3 can be substituted into Equation 7, and the risk is:

$$R_n = \sum_{x=0}^{n} \int_0^1 (E(t|x) - t)^2 \, g(t) \, h(x) \, dt. \qquad [8]$$

In the situation in which x and t are funtionally dependent, say $t = \phi(x)$, the conditional distribution of t given x degenerates to a single value of t given x. Therefore, when t is functionally depen-

dent on x, $E(t|x)$ will be equal to $t = \phi(x)$, and

$$R_c = \sum_{x=0}^{n} \int_0^1 (\phi(x) - t)^2 \, k(t,x) \, dt = 0. \qquad [9]$$

Substituting Equations 7, 8, and 9 into Equation 1 yields the coefficient δ for the regression function $E(t|x)$. Computation of δ requires that the regression function $E(t|x)$, and the probability densities $k(t,x)$, $g(t)$, and $h(x)$ can be estimated; we do not know any psychometric theory for estimating these functions simultaneously without assumptions about the form of the regression function.

Using Kelley's linear regression function

$$E_\ell(t|x) = \rho_{xx'}(x/n) + (1 - \rho_{xx'}) E(x/n). \qquad [10]$$

as a point estimator for t implies that the classical test model and the linearity of $E(t|x)$ are taken for granted (Lord & Novick, pp. 64-65). The risk that is connected to the estimator Equation 10 and loss function Equation 6 is:

$$E\,[E_\ell(t|x) - t]^2. \qquad [11]$$

This is the definition of the squared standard error of estimation:

$$\sigma_e^2 = \sigma_t^2 (1 - \rho_{xx'}) \qquad [12]$$

(Lord & Novick, p. 67). So the general form of the risk is:

$$R = \sigma_t^2 (1 - \rho_{xx'}). \qquad [13]$$

Stochastical independence of t and x implies linear stochastical independence:

$$\rho_{xt} = \rho_{xx'} = 0. \qquad [14]$$

Therefore, in the situation wherein t and x are stochastically independent Equation 13 has the form:

$$R_n = \sigma_t^2. \qquad [15]$$

It was remarked earlier that for an unspecified regression function Equation 9 applies when t and x are functionally dependent. So in the case of the linear regression function (Equation 10), the same result is valid for R_c:

$$R_c = 0. \qquad [16]$$

Substituting Equations 13, 15, and 16 into Equation 1 yields:

$$\delta = (\sigma_t^2 - \sigma_t^2 (1 - \rho_{xx'}))/\sigma_t^2 = \rho_{xx'} \qquad [17]$$

Therefore, in estimating the true score with Kelley's formula, the well-known reliability coefficient is interpretable as a standardized risk; its standardization is given by Equation 1.

Multiple Decisions

The true score is defined on the closed interval from 0 to 1. It is assumed that this interval is divided into (k+1) disjoint intervals: $[0, d_1)$, $[d_1, d_2)$, ... $[d_k, 1]$; these intervals are denoted as D_0, D_1, ..., D_k. The observed score x is defined on the set of integers: 0, 1, ... n. It is assumed that this set is divided into the disjoint sets: $\{0, 1, \ldots c_1-1\}$, $\{c_1, c_1+1, \ldots c_2-1\}$, ... $\{c_k, c_k+1, \ldots n\}$; these sets are denoted as C_0, C_1, ... C_k. In multiple decision situations the problem is to find cutting scores that, for a given loss function, optimally divide x into the sets C_i. For multiple decision situations it seems to make sense to split the loss function L(t,x) into $(k + 1)^2$ separate continuous functions for each combination of an interval D_i and a set C_j:

$$L(t,x) = \begin{cases} L_{00} & t \in D_0,\ x \in C_0 \\ \vdots & \\ L_{0k} & t \in D_0,\ x \in C_k \\ L_{10} & t \in D_1,\ x \in C_0 \\ \vdots & \\ L_{1k} & t \in D_1,\ x \in C_k \\ L_{20} & t \in D_2,\ x \in C_0 \\ \vdots & \\ L_{kk} & t \in D_k,\ x \in C_k \end{cases} \qquad [18]$$

Here, it will be required that the function L_{ij} is a continuous function that depends only on t for each combination of an interval D_i and a set C_j. The meaning is that the loss function for a true score from D_i and an observed score from C_j only depends on the true score. From a decision theoretic point of view this makes sense: the same decision is made for all observed scores from C_j; therefore, within C_j the loss function L_{ij} should not depend on the observed score but only on the true score.

Using the notation Σ_{C_j} for summation over the values of the observed score in the set C_j and $\int_{D_i}$ for the integral over the true score in the interval D_i the general form of the risk for multiple decision is:

$$R = E\, L(t,x) = \sum_{i=0}^{k} \sum_{j=0}^{k} \int_{C_j} \int_{D_i} L_{ij}(t)\, k(t,x)\, dt. \qquad [19]$$

In applications the loss functions $L_{ij}(t)$ should be specified further. An example of a possible set of functions that seems appropriate for multiple decisions is:

$$L_{ij}(t) = \ell_{ij} \qquad \begin{array}{l} \ell_{00} \leq \ell_{01} \leq \cdots\cdots\cdots\cdots\cdots \leq \ell_{0k} \\ \ell_{10} \geq \ell_{11};\ \ell_{11} \leq \ell_{12} \cdots\cdots\cdots \leq \ell_{1k} \\ \ell_{20} \geq \ell_{21} \geq \ell_{22};\ \ell_{22} \leq \ell_{23} \leq \cdots \leq \ell_{2k} \\ \ell_{k0} \geq \ell_{k1} \geq \cdots\cdots\cdots\cdots\cdots \geq \ell_{kk}. \end{array} \qquad [20]$$

The loss is constant for each combination of D_i and C_j. The loss is smallest for the combinations D_i and C_i, and increases when C_j is "further away" from D_i. Using the notation P_{ij} for the probability that the true score is in the interval D_i and the observed score in the set C_j the risk is for this loss function:

$$R = \sum_{i=0}^{k} \sum_{j=0}^{k} \ell_{ij}\, P_{ij}. \qquad [21]$$

Using the notation $P_{\cdot j} = \sum_{i=0}^{k} P_{ij}$ and $P_{i\cdot} = \sum_{j=0}^{k} P_{ij}$ it follows from Equation 3 that

$$R_n = \sum_{i=0}^{k} \sum_{j=0}^{k} \ell_{ij} P_{.j} P_{i.} \qquad [22]$$

The expression for R_c is more complicated. Whenever t has a functional relation to x, the bivariate distribution of t and x is degenerate with no scatter about the regression of t on x. Graphically, the functional relation $t = \phi(x)$ is a graph that contains points of the original space t x x from which the multiple decision table has been constructed by simultaneous partitioning along the t-axis, and x-axis. To every cell (i,j) of this table corresponds a subset $D_i \times C_j = [d_i, d_{i+1}) \times [c_j, c_{j+1}-1)$ of the original space. Dependent on the functional form of $t = \phi(x)$ and the cutting scores d_i and c_j, a cell (i,j) may or may not contain points of $t = \phi(x)$. Only when a cell (i,j) contains points of $t = \phi(x)$ can it have a non-zero probability of occurrence and a non-zero contribution to the risk. In the case of functional dependence between t and x the following is true for the bivariate density:

$$k(t,x) = h(x). \qquad [23]$$

Therefore, the contribution of cell (i,j) to the risk R_c can be written as:

$$r_{ij} = \sum_{S_{ij}} \ell_{ij} h(x). \qquad [24]$$

where S_{ij} is the set of all values of x for which $(x,\phi(x))$ is a point of $D_i \times C_j$. Summation over all cells of the decision table yields:

$$R_c = \sum_{i=0}^{k} \sum_{j=0}^{k} r_{ij} \qquad [25]$$

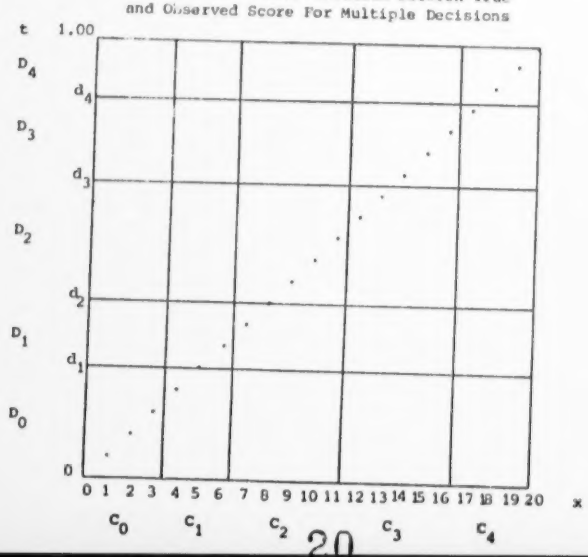
and substituting Equations 21, 22, and 25 into Equation 1 gives the decision coefficient δ for multiple decisions with loss function Equation 20 and a regression function as yet unspecified. In situations in which the linear regression function Equation 10 can be assumed, $t = \phi(x)$ equals x/n in the case of functional dependence, and the sets S_{ij} are easily obtained. S_{ij} is the set of all values of x for which $(x,x/n)$ is an element of $D_i \times C_j$, meaning that

$$S_{ij} = [nd_i, nd_{i+1}) \cap (c_j, c_{j+1}-1) \qquad [26]$$

(the interval $[d_i, d_{i+1})$ has been multiplied by n to give t and x the same scale). Figure 1 clarifies Equation 26:

Figure 1

A Linear Functional Relation between True and Observed Score For Multiple Decisions



All probability mass is concentrated on the points representing the linear relation $t = x/n$. The sets S_{ij}, containing all values of x for which $(x, x/n)$ is a point of cell (i,j), is exactly the intersection of the sets $[nd_i, nd_{i+1})$ and $\{c_j, c_{j+1}\}$.

Assuming the beta-binominal model, which implies that $h(x)$ has the negative hypergeometric form and that the regression of t on x is linear (Lord & Novick, 1968, Chap. 23), $h(x)$ and the probabilities P_{ij}, $P_{i.}$, and $P_{.j}$, which are needed for the computation of δ, can be estimated in the same way as for dichotomous decisions (Koppelaar, Van der Linden, & Mellenbergh, 1977). Although δ can be computed under these assumptions the coefficient is not necessarily between 0 and 1. The reason is that it depends on the loss function Equation 20 and on the bivariate distribution $k(t,x)$ whether or not the risk R is in the interval from R_c to R_n.

Van der Linden and Mellenbergh (1977) described a linear loss function for dichotomous decisions. A generalization of this function seems suitable for multiple decisions; the loss is a linear function of t for each cell of the decision table:

$$
\begin{aligned}
L_{i0}(t) &= b_0(t - d_1) + a_0 && i = 0,1, \ldots\ldots\ldots k \\
L_{ij}(t) &= \begin{cases} b_j(d_j - t) + a_j & t \le t_j \\ b_j'(t - d_{j+1}) + a_j' & t > t_j \end{cases} && i = 0,1, \ldots\ldots\ldots k \\
& && j = 1,2, \ldots (k-1) \\
& && t_j \in (d_j, d_{j+i}) \\
L_{ik}(t) &= b_k(d_k - t) + a_k && i = 0,1, \ldots\ldots\ldots k
\end{aligned}
\qquad [27]
$$

An example will clarify some of the statements made above. Suppose that in achievement measurement the true score interval is divided into five categories:

$\underline{D}_0$: [0, 0,40], "very bad";

$\underline{D}_1$: [0.40, 0.60], "insufficient";

$\underline{D}_2$: [0.60, 0.75], "sufficient";

$\underline{D}_3$: [0.75, 0.90], "good";

$\underline{D}_4$: [0.90, 1.00], "excellent".

A twenty-five item four-choice test is administered. The observed scores are divided into five categories and grades are assigned as follows:

$\underline{C}_0$: {0,1, ... 14} , grade E;

$\underline{C}_1$: {15, 16, 17} , grade D;

$\underline{C}_2$: {18, 19, 20} , grade C;

$\underline{C}_3$: {21, 22, 23} , grade B;

$\underline{C}_4$: {24, 25} , grade A.

A specification of the loss function Equation 20 is reported in Table 1; hypothetical proportions $P_{i.}$ and $P_{.j}$ are also reported in the table. Using Equation 22 R_n can be calculated for these hypothetical data. A specification of the loss function Equation 27 is illustrated in Table 2.

Table 1
A specification of the Loss Function Equation 20 with Hypothetical Proportions $P_{i.}$ and $P_{.j}$

	Observed Score					
True Score	C_0	C_1	C_2	C_3	C_4	$P_{i.}$
D_0 .40	0	1	2	3	4	.10
D_1 .60	1	0	1	2	3	.20
D_2 .75	2	1	0	1	2	.40
D_3 .90	3	2	1	0	1	.20
D_4	4	3	2	1	0	.10
$P_{.j}$	.20	.10	.50	.15	.05	

Table 2

A specification of the Loss Function Equation 27 with $a_2 = a_3 = a_4 = a'_2 = a'_3 = .02$, $a_0 = a_1 = a'_1 = .10$, $b_0 = b_4 = 1$, $b_j = b'_j = 1$ $(j=1,2,3)$

True Score	Observed Score C_0	C_1	C_2	C_3	C_4
D_0	t-.30	.50-t	.62-t	.77-t	.92-t
.40					
D_1	t-.30	.50-t / t-.50	.62-t	.77-t	.92-t
.60					
D_2	t-.30	t-.50	.62-t / t-.73	.77-t	.92-t
.75					
D_3	t-.30	t-.50	t-.73	.77-t / t-.88	.92-t
.90					
D_4	t-.30	t-.50	t-.73	t-.88	.92-t

In Table 2 the following specification is chosen for a_j : $a_2 = a_3 = a_4 = a_2' = a_3' = 0.02$ representing, for instance, the cost for testing, and $a_0 = a_1 = a_1' = 0.10$ representing, for instance, the cost for testing and for remedial teaching; for b_i the specification is chosen: $b_0 = b_4 = 1$, $b_j = b_j' = 1$ $(j = 1,2,3)$.

Dichotomous Decisions

If $k = 1$ multiple decisions reduce to dichotomous decisions. The true score t is divided into two disjoint intervals $D_0 = [0,d)$ and $D_1 = [d,1]$, and the observed score x into the two sets $C_0 = \{0,1,\ldots\ c-1\}$ and $C_1 = \{c,c+1,\ldots\ n\}$ (see Table 3). Given a cutting score c that is optimal for a defined loss function, the best decision for an observed score from the set C_i is to assume that the true score is in the interval D_i.

Table 3

Dichotomous Decisions with Proportions

	Observed Score		
True Score	$x < c$ (c)	$x \geq c$	$P_{i.}$
$t < d$ (d)	P_{00}	P_{01}	$P_{0.}$
$t \geq d$	P_{10}	P_{11}	$P_{1.}$
	$P_{.j}$ $P_{.0}$	$P_{.1}$	

Loss function Equation 20 has the following form for dichotomous decisions:

$$L_{ij}(t) = \begin{cases} \ell_{00} & \text{for} \quad t < d, \; x < c \\ \ell_{10} & \text{for} \quad t \geq d, \; x < c \\ \ell_{01} & \text{for} \quad t < d, \; x \geq c \\ \ell_{11} & \text{for} \quad t \geq d, \; x \geq c \end{cases} \qquad [28]$$

where $\ell_{00} \leq \ell_{01}$ and $\ell_{10} \geq \ell_{11}$. The coefficient δ for dichotomous decisions and this loss function results from Equations 28, 25, 24, 22, 21, and 1 and the specification $k = 1$. The restrictions $\ell_{00} = \ell_{11} = 0$ and $\ell_{10} = \ell_{01} = \ell$ for loss function Equation 28 together with the assumption of an increasing regression function of t on x, give rise to an interesting result. The loss function represents the situation in which the loss is equal for both correct classifications as well as for both misclassifications, and in which a rescaling has taken place to give both correct decisions a zero loss; the risk is:

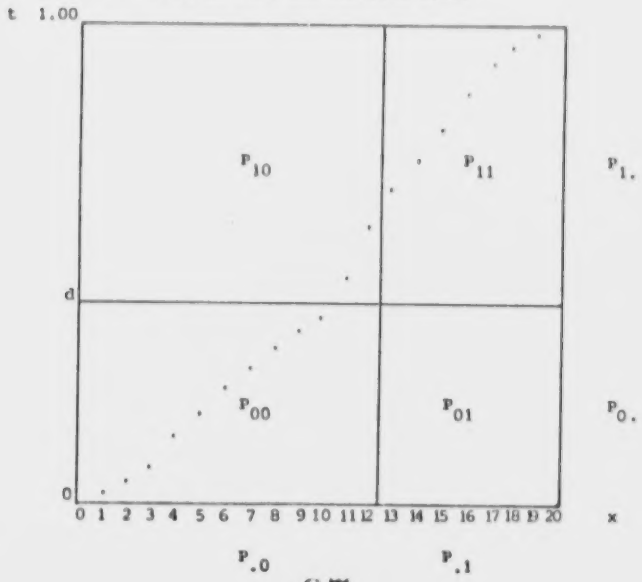
$$R = (P_{01} + P_{10})\, \ell. \qquad [29]$$

In case of stochastical independence between t and x the risk is:

$$R_n = (P_{0.}\, P_{.1} + P_{1.}\, P_{.0})\, \ell. \qquad [30]$$

For an increasing regression function of t on x, and for functional dependence of t and x the true score t is an increasing function of the observed score x. It follows that P_{11} is the smallest of the probabilities $P_{1.}$ and $P_{.1}$, and that either $P_{01} = 0$ (for $P_{.1} \leq P_{1.}$) or $P_{10} = 0$ (for $P_{.1} > P_{1.}$). For instance, when $P_{1.} \geq P_{.1}$, then $P_{11} = P_{.1}$ and $P_{01} = 0$; also, $P_{10} = P_{1.} - P_{.1}$, and $P_{00} = P_{0.}$. This is demonstrated in Figure 2. All probability mass is concentrated on the points representing the increasing functional relation between t and x. The probability P_{01} is equal to 0; the probability P_{11} is equal to the probability $P_{.1}$, the smallest of the probabilities $P_{1.}$ and $P_{.1}$.

Figure 2
An Increasing Functional Relation
Between True and Observed Score



Therefore, R_c can be written as:

$$R_c = \begin{cases} (P_{1.} - P_{.1})\,\ell & \text{for } P_{1.} \geq P_{.1} \\ (P_{.1} - P_{1.})\,\ell & \text{for } P_{1.} < P_{.1} \end{cases} \qquad [31]$$

Substituting Equation 29, 30, and 31 into Equation 1, and simplifying yields:

$$\delta = \begin{cases} (P_{11} - P_{1.}\,P_{.1})/(P_{.1}(1 - P_{1.})) & \text{for } P_{1.} \geq P_{.1} \\ (P_{11} - P_{1.}\,P_{.1})/(P_{1.}(1 - P_{.1})) & \text{for } P_{1.} < P_{.1} \end{cases} \qquad [32]$$

This is the well-known coefficient H of Loevinger! The coefficient can be estimated in case the proportions P_{ij} can be estimated. Assuming the aforementioned beta-binomial model, this can be done. Furthermore, the same derivation applies of course for a linear regression function of t on x because the linear regression function is an increasing function of x. In this case, however, the same result can be obtained from the general result using Equation 26.

For dichotomous decisions loss function Equation 27 takes the following form:

$$L(t) = \begin{cases} b_0(t - d) + a_0 & \text{for } x < c \\ b_1(d - t) + a_1 & \text{for } x \geq c \end{cases} \qquad [33]$$

A simple formula for computing the optimal value of the cutting score c, given this linear function, has been derived; the risk is in this case (Van der Linden, & Mellenbergh, 1977):

$$R = \sum_{x=0}^{c-1} [b_0(E(t|(x/n)) - d) + a_0]h(x) - \sum_{x=c}^{n} [b_1\,(E(t|(x/n)) - d) + a_1]h(x). \qquad [34]$$

The assumption of the linear regression function of t on x (Equation 10) also gives rise to an interesting result. Substituting Equation 10 into Equation 34 gives the risk R. In the situation in which t and x are stochastically independent, there is also linear stochastical independence, and Equation 10 has the form:

$$E_I(t|x) = E(x/n). \qquad [35]$$

Substituting Equation 35 into Equation 34 gives the risk R_n for stochastical independence between t and x. It was shown earlier that for the situation in which t and x are functionally dependent, there is also linear dependence, and Equation 10 has the form:

$$E_I(t|x) = x/n. \qquad [36]$$

Substituting Equation 36 into Equation 34 gives the risk R_c for functional dependence between t and x. The result of substituting R, R_n, and R_c into Equation 1, and straightforward algebra is simply:

$$\delta = \rho_{xx'}. \qquad [37]$$

Therefore, in the case of a linear regression function of t on x, and the linear loss function Equation 33, the well-known reliability coefficient is interpretable as a standardized risk; its standardization is given by Equation 1.

Discussion

Attempts have recently been made to construct coefficients for dichotomous decision situations (Livingston, 1972; Harris, 1974). These attempts are not based on a theoretical foundation such as the decision theoretic point of view presented here. The results of this development were rather surprising: two old friends of psychometrics have shown up again. First, Loevinger's coefficient H between the dichotomized true and observed score for loss function Equation 28. Second, the reliability coefficient in the case of loss function Equation 33, and Kelley's formula for estimating the true score. An obvious disadvantage of Equation 28 is that the loss is constant for each cell of the twofold table. For instance, a not-accepted subject with a true score just above the cutting scores gives the same loss as a not-accepted subject with a true score far above the cutting score. Therefore, in most applications Equation 33 will be more realistic than Equation 28 for expressing the losses in dichotomous decision situations. Our feeling is that there is not much need for a new coefficient specially adapted to dichotomous decision situations. In many practical applications the reliability coefficient will be sufficient from a decision theoretic point of view.

Using loss function Equation 28, Huyhn (1976a) has described a coefficient c for dichotomous decision situations. The coefficient is derived in a way that is concistent with our approach: it is also

constructed from a decision theoretic point of view. The coefficient is defined as:

$$\varepsilon = (R^{*} - R_0)/R^{*}. \qquad [38]$$

Comparing Equation 38 with Equation 1 it appears that Huyhn uses R_0 instead of R, and R^{*} instead of R_n. R_0 is the risk for the situation in which an optimal cutting score is used, whereas R is the risk when an arbitrary cutting score is used. In practical applications a cutting score is often fixed before the test is administered, and this fixed cutting score is in general not optimal. Hence the use of R instead of R_0 seems to be more adapted to practical situations; furthermore, R is more general because R equals R_0 if the optimal cutting score is applied. Deriving R^{*} Huyhn starts with the independence of t and x; so far his derivation is identical to R_n (Equation 22) with loss function Equation 28. In this case Huyhn also looks for an optimal cutting score. He observes that the minimum risk accompanying a dichotomous decision, loss function Equation 28, and stochastical independence between t and x is attained for either $c = c_f$ (the cutting score for which all subjects fail) or $c = c_p$ (the cutting score for which all subjects pass). Therefore R^{*} is defined as the minimum value of the risk for either $c = c_f$ or $c = c_p$. Hence R^{*} is the risk for the situation in which the true and observed scores are distributed independently, and in which either all subjects fail the test or pass the test. R_n is derived for the situation in which the true and observed scores

are distributed independently, and in which, on the contrary, the chosen cutting score has been maintained. Like K this seems to be better adapted to practical situations. Another difference between the coefficients ε and δ is that ε is only defined for dichotomous decisions with loss function Equation 28, whereas δ is defined for a broad class of decision situations and loss functions. In our opinion, therefore δ seems to be a more realistic and more general coefficient than ε.

In the literature on criterion-referenced measurement or mastery testing the use of coefficient kappa (Cohen, 1960) for determining the reliability of decisions is becoming popular (Swaminathan, Hambleton, & Algina, 1974). Coefficient kappa is computed for data collected according to a test-retest or test-parallel test design, and reflects the chance corrected consistency of decisions. Here, interest is not in determining consistency of decisions, but in deriving a coefficient that represents the optimallity of decisions. Therefore, instead of a test-retest or test-parallel test design the decision table is considered, which has the latent and observed scores along its axes. For this table a coefficient like Cohen's kappa is not only less natural than a coefficient based on decision theoretic risk, but is even erroneous. Coefficient kappa is introduced by Cohen (1960) as a coefficient that expresses the degree of agreement between two nominal scales corrected for chance. Moreover, it should be noted, that computing kappa supposes equal marginal distributions for both nominal scales, because in case of differing marginal distributions kappa can not reach its maximal value. However, for the decision table equal marginal distributions are not necessary at all: The used cutting scores, and the unreliability of the observed score will give rise to differing marginal distributions.

Furthermore, determining the optimallity of decisions is not a matter of agreement between nominal scales, but of association between the ordered categories along both axes of the decision table. Not only the for chance corrected proportion of cases in the diagonal, but all combinations of the true and observed score should be considered, each weighted according to the used loss function. Though weighted coefficient kappa (Cohen, 1968) makes weighing procedures possible, it is inappropriate as well. With this coefficient only weighing by loss functions of type Equation 18 is possible. More plausible loss functions like Equations 27 and 33 can not be used for this purpose. On the contrary, an attractive feature of coefficient δ is that it can be derived for decision procedures with loss functions of any type.

Finally, doe notethree points. One, it is emphasized that δ is not always in the interval from 0 to 1. It has been shown that the coefficient is in this interval for some important applications, for example at least coefficient H and the reliability coefficient are in the interval. Two, at several places the beta-binomial model is mentioned. But whether this model fits the data should be investigated; moreover, the statistical properties of estimates computed under the assumption of this model are not known. Three, there seems not to be much interest in multiple decision situations; emphasis has mainly been on point estimation and dichotomous decisions. Many situations, however, are in fact multiple decision situations. For instance, subjects with scores equal or above a score c_1 on a test pass the test, subjects with scores below c_0 fail the test, and subjects with scores between c_0 and c_1 are retested or get special treatments such as remedial teaching. Therefore, multiple decisions require attention.

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